

Hawking Radiation from Feynman Diagrams

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Abstract

The aim of this letter is to clarify the relationships between Hawking radiation and the scattering of light by matter falling into a black hole. To this end we analyze the S-matrix elements of a model composed of a massive infalling particle (described by a quantized field) and the radiation field. These fields are coupled by current-current interactions and propagate in the Schwarzschild geometry. As long as the photons energy is much smaller than the mass of the infalling particle, one recovers Hawking radiation since our S-matrix elements identically reproduce the Bogoliubov coefficients obtained by treating the trajectory of the infalling particle classically. But after a brief period, the energy of the ‘partners’ of Hawking photons reaches this mass and the production of thermal photons through these interactions stops. The implications of this result are discussed.

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I. INTRODUCTION

Upon deriving black hole radiance, Hawking [1] found that ω , the frequency of the in-modes involved in the Bogoliubov coefficients, grows exponentially according to

$$\omega = \lambda e^{\kappa(u-u_0)} \quad (1)$$

where κ is the surface gravity of the hole and where u is the retarded time around which the out-particle of energy λ is centered. This point was emphasized by Gerlach [2] (and subsequently in [3]) who showed that the constant emission rate arises from a steady conversion of vacuum configurations of frequency ω into red-shifted on shell photons of energy λ .

This observation questions the validity of the settings used to derive Hawking radiation, namely free field propagation in a given geometry. Indeed, knowing that gravitational interactions grow with the energy, what is the validity of describing these high frequency configurations by free field theory [4–7]. This question becomes particularly important when one considers the interactions between the collapsing matter and Hawking quanta. In this respect, what is well understood is that if one first solves Einstein’s equations to determine the collapsing metric and then study free field propagation in this geometry, one obtains Hawking radiation through the frequency mixing described in eq. (1). In this derivation, the motion (and the quantum state) of infalling matter is unaffected by the emission of Hawking quanta. Indeed, the configurations giving rise to these quanta freely propagate through the collapsing matter [1,7].

The aim of this article is to provide a more dynamical description of the interactions between infalling matter and the radiation field ϕ . To this end we analyze a simple model composed of ψ , a field of mass m , coupled to ϕ by current-current interactions and propagating in Schwarzschild geometry. Thus the quanta of ψ represent additional particles falling into an already formed black hole. Since the interactions are described by Feynman diagrams, the infalling particles are now properly scattered according to energy-momentum conservation. This is crucial since it reveals the dynamical role played by ω , the ‘trans-planckian’ energy of the ‘partner’ of an asymptotic quantum of energy λ . Indeed, we now

obtain two different regimes. First, as long as ω is much smaller than m , the mass of the infalling dust particle, the S-matrix elements of our model identically reproduces the Bogoliubov coefficients obtained by attributing a given inertial trajectory to the infalling particle. In this regime one thus recovers the infalling mirror description of Hawking radiation [8–12] with one important improvement: the residual energy crossing the future horizon is equal to the energy of the infalling particle minus the energy carried away by the Hawking quanta. Therefore, one does not need to appeal to Einstein’s equations in order to obtain the notion of black hole evaporation.

The second regime occurs when ω becomes comparable to m . Then the scattering amplitudes no longer agree with those found by Hawking. In fact, because of energy conservation, we shall see that the production of thermal photons (induced by scattering on the infalling particle) stops when ω reaches m . With this result, we reach the heart of the problem: how to obtain a steady conversion of vacuum configurations giving rise to a constant thermal flux once recoil effects are no longer neglected. The questions raised by our negative result are addressed at the end of the letter.

It should also be mentioned that other derivations of Hawking radiation do not confront the transplanckian problem in those radical terms. First, superspring theory succeeded in deriving Hawking radiation in completely different settings since it results from the degeneracy of black hole microstates [13]. Moreover, the derivation is performed in flat space. Thus one confronts neither exponentially growing Doppler effects, which are the all mark of regular horizons, nor therefore the high frequencies.

Secondly, phenomenological models based on analogies with condensed matter physics have been proposed in order to question the relevance of these high frequencies [14,15]. At present however, one does not know how to justify dynamically these rather ad hoc models. Perhaps the present approach can provide the roots for such a justification.

II. THE MODEL

In this letter, for simplicity, we shall consider only radial motion. Therefore, we can work with 2 dimensional fields. It is then convenient to use the conformally flat coordinates t, z in which the 2D line element reads

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) (-dt^2 + dz^2) \\ &= \left(1 - \frac{2M}{r}\right) \eta_{\mu\nu} dx^\mu dx^\nu \end{aligned} \quad (2)$$

where $\eta^{\mu\nu}$ is the 2D Minkowski unit matrix and z the tortoise coordinate ($= r + 2M \ln(r/2M - 1)$).

In this coordinate system, the action of the system is

$$\begin{aligned} S &= \int dt dz \left(-\eta^{\mu\nu} \partial_\mu \psi^* \partial_\nu \psi - \left(1 - \frac{2M}{r}\right) m^2 |\psi|^2 \right) \\ &\quad + \int dt dz \left(-\eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - g \eta^{\mu\nu} J_\mu^\psi J_\nu^\phi \right) \end{aligned} \quad (3)$$

where $J_\mu^\psi = \psi^* \overleftrightarrow{\partial}_\mu \psi$, $J_\nu^\phi = \phi^* \overleftrightarrow{\partial}_\nu \phi$ are the currents carried by the complex fields ψ and ϕ . g is the coupling constant. We have chosen to work with two complex fields in order to have well defined current operators before splitting positive and negative frequencies, see [16] for more details concerning this model.

In the absence of interactions ($g = 0$), the modes of the fields freely propagate in Schwarzschild geometry. Since the metric is static, they can be labeled by their constant energy. The massless modes describing infalling and outgoing photons are respectively

$$\begin{aligned} \phi_\omega &= \frac{e^{-i\omega(t+z)}}{\sqrt{4\pi\omega}} = \frac{e^{-i\omega_\mu x^\mu}}{\sqrt{4\pi\omega}} \\ \phi_\lambda &= \frac{e^{-i\lambda(t-z)}}{\sqrt{4\pi\lambda}} = \frac{e^{-i\lambda_\mu x^\mu}}{\sqrt{4\pi\lambda}} \end{aligned} \quad (4)$$

In the WKB approximation, the infalling mode of the massive field with energy ϵ is

$$\psi_\epsilon = \frac{e^{-i(\epsilon t + \int^z dz' p_\epsilon)}}{\sqrt{4\pi p_\epsilon(z)}} \quad (5)$$

where $p_\epsilon(z)$ is the classical momentum, the positive solution of the mass-shell condition

$$\eta^{\mu\nu} p_\mu p_\nu = -\epsilon^2 + p_\epsilon^2 = -m^2(1 - 2M/r) \quad (6)$$

For $m \gg \kappa$ the WKB approximation is valid if one does not approach the turning point $p_\epsilon(z) = 0$ which exists when $\epsilon < m$.

In the interacting picture, to first order in g , the transition amplitudes are given by the matrix elements of $g \int dt dz J_\mu^\psi J_\phi^\mu$. Denoting $A(\epsilon; \omega, \lambda)$ the amplitude for an infalling photon of energy ω to be scattered by a dust particle of energy ϵ and converted into an outgoing photon of energy λ , one gets

$$A(\epsilon; \omega, \lambda) = -ig \int dz (p_\mu(\epsilon) + p_\mu(\epsilon + \omega - \lambda)) (\omega^\mu + \lambda^\mu) \times \frac{e^{-iz(\lambda+\omega)} e^{-i \int^z dz' (p_\epsilon - p_{\epsilon+\omega-\lambda})}}{4\pi\sqrt{\lambda\omega} \quad 2\sqrt{p_\epsilon p_{\epsilon+\omega-\lambda}}} \quad (7)$$

where the final energy of the particle is $\epsilon + \omega - \lambda$ since energy conservation is implemented by the integration over t . The prefactor in the first line of eq. (7) comes from the matrix elements of the two current operators.

By crossing symmetry, the amplitude to spontaneously produce these two photons of energy ω and λ from scattering by an infalling particle of energy ϵ is given by

$$B(\epsilon; \omega, \lambda) = A(\epsilon; -\omega, \lambda) \quad (8)$$

As we shall see, it is through these Bremsstrahlung-like amplitudes that Hawking radiation will be recovered in the low energy regime.

III. LOW ENERGY REGIME

In this regime, i.e. for $\omega, \lambda \ll m$, one can develop the phase and the prefactor of the integrand of $A(\epsilon; \omega, \lambda)$ in powers of ω and λ . To first order, we get

$$A(\epsilon; \omega, \lambda) \simeq -ig \int dz [(\omega + \lambda) dt_{cl}/dz - (\omega - \lambda)] \times \frac{e^{-iz(\lambda+\omega)} e^{-it_{cl}(z)(\omega-\lambda)}}{4\pi\sqrt{\lambda\omega}} \quad (9)$$

where

$$t_{cl}(z) = -\partial_\epsilon \int^z dz' p_\epsilon(z') \quad (10)$$

is the time lapse evaluated along the infalling particle trajectory. We also used $\epsilon/p_\epsilon(z) = dt_{cl}/dz$.

To first order in ω, λ , we find that $A(\epsilon; \omega, \lambda)$ is proportional to $\alpha(\omega, \lambda)$, the overlap of the photons wave functions evaluated along the classical trajectory of the infalling particle characterized by ϵ and parametrized by z through $t_{cl}(z)$. This is easily verified by direct computation. Similarly, one finds that the amplitude $B(\epsilon; \omega, \lambda)$ is proportional, with the *same* factor, to $\beta(\omega, \lambda) = \alpha(-\omega, \lambda)$, the Bogoliubov coefficient encoding pair creation. It should be stressed that these agreements to first order in the energy-momentum transfers, here given by ω and λ , are generic in character, see [17].

We now focus on the late time regime, for $r/2M - 1 \simeq e^{2\kappa z} \ll 1$. In this case one has [2,3]

$$\frac{dt_{cl}}{dz} = -1 - e^{2\kappa(z-z_0)} \quad (11)$$

Then upon applying the stationary phase condition to the integrand of eq. (9) one recovers eq. (1) since in the late time regime $z = -u/2 + \text{constant}$ along the infalling trajectory. This confirms that

$$\left| \frac{B(\epsilon; \omega, \lambda)}{A(\epsilon; \omega, \lambda)} \right|^2 = \left| \frac{\beta(\omega, \lambda)}{\alpha(\omega, \lambda)} \right|^2 = e^{-2\pi\lambda/\kappa} \quad (12)$$

Eq. (12) establishes the thermal character of the outgoing photons spontaneously emitted by the scattering on the infalling particle. As usual ‘spontaneously’ means that one starts from vacuum configurations on \mathcal{I}^- .

It is now appropriate to show how the notion of a constant rate occurs. To order g^2 , the mean number of photons of energy λ found on \mathcal{I}^+ , is given by

$$\langle n_\lambda \rangle = \int d\omega |B(\epsilon; \omega, \lambda)|^2 \quad (13)$$

To first order in ω, λ it is proportional to $\int d\omega/\omega$, as in [3,18]. To give meaning to this integral, one uses the fact that at time u , the λ photons arise from frequencies ω centered

according to eq. (1). This yields $d\omega/\omega = \kappa du$, i.e. that the production rate is constant. Similarly, the mean number of photons received before a certain time u , is obtained by integration over ω up to $\lambda e^{\kappa(u-u_0)}$. Thus it increases linearly with $u - u_0$. This establishes that the thermal flux originates from a steady conversion of vacuum configurations into pairs of photons whose energy are related by eq. (1).

Thus, in this linear approximation in ω, λ , one recovers Hawking radiation as obtained from free field theory. Moreover, in our description based on matrix elements, the *residual* energy which crosses the future horizon is equal to ϵ minus the energy of the outgoing λ quanta since energy is conserved and since the ω quanta fall into the hole and do not enter into this global energy balance. Therefore we obtain the notion of evaporation without having used Einstein's equations, but simply by having followed the standard rules of quantum field theory.

The most remarkable feature of these results is the steadiness of the production rate. However, as we shall see, it is a consequence of having performed a first order expansion in ω . Moreover, because of eq. (1), ω reaches m after a few e-folds in the units of $1/\kappa$. Therefore, since the low energy regime is brief, it is mandatory to take into account higher order effects in ω/ϵ .

IV. HIGH ENERGY REGIME

To characterize the high energy regime we first analyze the classical channel, i.e. the scattering of an infalling photon of frequency ω which is sent from \mathcal{I}^- . The simplest way to proceed consists in applying the stationary phase condition to the phase of the integrand of $A(\epsilon; \omega, \lambda)$. Then, the dominant contribution arises from values of z centered around the saddle point value z_{sp} , the solution of

$$\omega + \lambda = p_{\epsilon+\omega-\lambda}(z_{sp}) - p_{\epsilon}(z_{sp}) \quad (14)$$

In the late time regime and for $\omega \gg \lambda \simeq \kappa$, it obeys

$$e^{2\kappa z_{sp}} = \frac{4\lambda}{\omega} \frac{\epsilon^2}{m^2} \left(1 + \frac{\omega}{\epsilon}\right) \quad (15)$$

As long as $\omega/\epsilon \ll 1$ one recovers the low energy regime since eq. (15) is equivalent to eq. (1).

Instead, when ω approaches ϵ , one reaches a new regime: the maximal value of λ , the energy of the scattered photon found on \mathcal{I}^+ , is bounded by $me^{-\kappa(u-u_0)}/4$ no matter how ω big is. Thus one loses the illusionary possibility offered by eq. (1), of receiving, on \mathcal{I}^+ and at arbitrarily large times, photons with a given frequency λ by sending from \mathcal{I}^- quanta with sufficiently high frequencies ω .

In loosing this possibility, we recover conventional physics: a particle of mass m cannot properly reflect photons of energy (measured in its rest frame) higher than its mass, see eq. (62) in [16]. Then, because of the red shift from the scattering locus z_{sp} to \mathcal{I}^+ , one obtains the announced bound for λ . We believe that a similar conclusion should also emerge from taking into account the deformation of the geometry induced by a infalling wave of energy $\hbar\omega$. In both cases, eq. (1) would be invalidated after time lapses of the order of a few $1/\kappa$.

Having clarified the direct channel, we now turn to spontaneous pair creation amplitudes. Due to crossing symmetry, the stationary phase conditions applied to $B(\epsilon; \lambda, \omega)$ are given by eqs. (14) and (15) with ω replaced by $-\omega$.

As for the amplitude A , the new regime arises when ω approaches ϵ . To characterize the onset of this regime, one should determine the first order corrections in ω/ϵ . This is easily achieved by exploiting the following fact. Besides the exponential $e^{i(\omega-\lambda)z}$, the phase of the integrand of B is an anti-symmetric function in $\omega + \lambda$ when developed around the ‘mean’ energy $\bar{\epsilon} = \epsilon/2 - (\epsilon - \omega - \lambda)/2$. Therefore, there is no quadratic terms in ω when one develops around $\bar{\epsilon}$. (Note in passing that this result directly follows from quantum mechanics which dictates that the wave functions of the ‘heavy’ system enter transition amplitudes always in products of the form $\psi_{\epsilon_{fm}}^* \psi_{\epsilon_{in}}$, see [17,19].)

Thus the improved value of $B(\epsilon; \omega, \lambda)$ is given by the first order expression with ϵ replaced by $\bar{\epsilon}$. From this one deduces the following. Firstly, the first order correction to Hawking

temperature vanishes. Indeed, eq. (12) still obtains since the value of ϵ plays no role to first order in ω . Secondly, the emission rate at which these thermal quanta are emitted is modified. Indeed, upon computing their mean number, eq. (13), one faces a new phase space volume. We remind the reader that when dealing with eq. (1), one had $d\omega/\omega = \kappa du$ which guaranteed a constant flux. Instead, upon using eq. (15), one gets

$$\frac{d\omega}{\omega} \simeq \kappa du \left(1 - \frac{\omega}{\epsilon}\right) \quad (16)$$

which implies a decreasing rate when ω/ϵ approaches 1.

It is of little interest to further characterize this decrease because one encounters an unavoidable barrier: the final energy of the scattered particle ($= \epsilon - \omega - \lambda$) cannot be negative. Indeed, if one follows the conventional rules of QFT, wave functions with negative energy do not correspond to on shell particles and therefore do not appear in matrix elements at the tree approximation. Therefore, the high energy regime (whose unbounded character was at the origin of the constant rate in the usual derivation) is simply not available since the upper bound in the integral of eq. (13) is given by $\epsilon - \lambda$. Thus the steadiness of the emission rate is lost and the total number of quanta emitted is bounded. (Notice that a different result has been obtained in [12] from a somehow more kinematical model.)

V. CONCLUSIONS

Upon analyzing the scattering amplitudes $A(\epsilon; \omega, \lambda)$ and $B(\epsilon; \omega, \lambda)$ we found two regimes. As long as $\omega/\epsilon \ll 1$, it is legitimate to perform a first order expansion in ω/ϵ and one recovers the usual Bogoliubov coefficients. In fact this agreement defines the physical interpretation of these coefficients: they are the correct amplitudes as long as backreaction effects can be neglected. This is why they only depend on ω, λ and κ .

When ω approaches ϵ , after a few $1/\kappa$ time lapses, the thermal production stops. Indeed, because of energy conservation, one can no longer appeal to unbounded frequencies ω , as one does in the absence of backreaction.

In this we have reached our aim: to show when how and why a simple approach based on QFT fails to reproduce Hawking radiation at large times. Needless to say that it is over hasty to deduce from this that Hawking radiation actually stops. Rather it poses with greater accuracy the question: which fundamental theory is Hawking's derivation an approximation of ?

A first radical option consists in postulating that Hawking radiation *cannot* be recovered from QFT based on General Relativity once backreaction effects are included. Instead it should emerge from other settings such as string theory [13]. Then, our negative result can be considered as an indication in favour of this belief.

A less radical approach consists in hoping that one shall recover Hawking radiation by improving the present analysis. (Notice that this conservative approach does work for the Unruh effect [20]. In that case, the backreaction effects obtained from QFT are bounded and the corrections to the Unruh effect stay finite [21].)

The improvements can be pursued in two directions. Firstly, by including scattering on many dust particles, one might get collective effects which significantly differ from what we obtained. Secondly, by including loop corrections one could recover access to the infinite reservoir of high frequencies and find that Hawking radiation does not emerge from on-shell diagrams. Indeed, one should explore the consequences of modifying Feynman rules for the external legs which correspond to quanta falling into the hole since there is no a priori reason to treat asymptotic and infalling quanta on the same footings. In this respect one can already notice that, due to the r -dependence in eq. (6), the final energy of the scattered particle can be smaller than m even though the corresponding particle cannot be found asymptotically.

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